Lecture 6

Dot Products and Projections

DSC 40A, Spring 2024

Announcements

- Homework 2 is due **tonight**. Remember that using the Overleaf template is required for Homework 2 (and only Homework 2).
- Check out the new FAQs page and the tutor-created supplemental resources on the course website.
    - The proof that we were going to cover last class (that $R_{\text{sq}}(w_0^*, w_1^*) = \sigma_y^2(1 - r^2)$) is now in the FAQs page, under Week 3.

Today, 5-6 PM
HDSI 123
panel about the capstone program!

DSC Undergraduate Town Hall

Monday, April 22nd, 1-3PM

HDSI 123

Come ask questions and voice your feedback about the undergraduate program, while socializing with faculty!

Your favorite professors will be there – and so will free cookies! 🍪🥠

Scan me to RSVP!

Agenda

- Recap: Friends of simple linear regression.
- Dot products.
- Spans and projections.

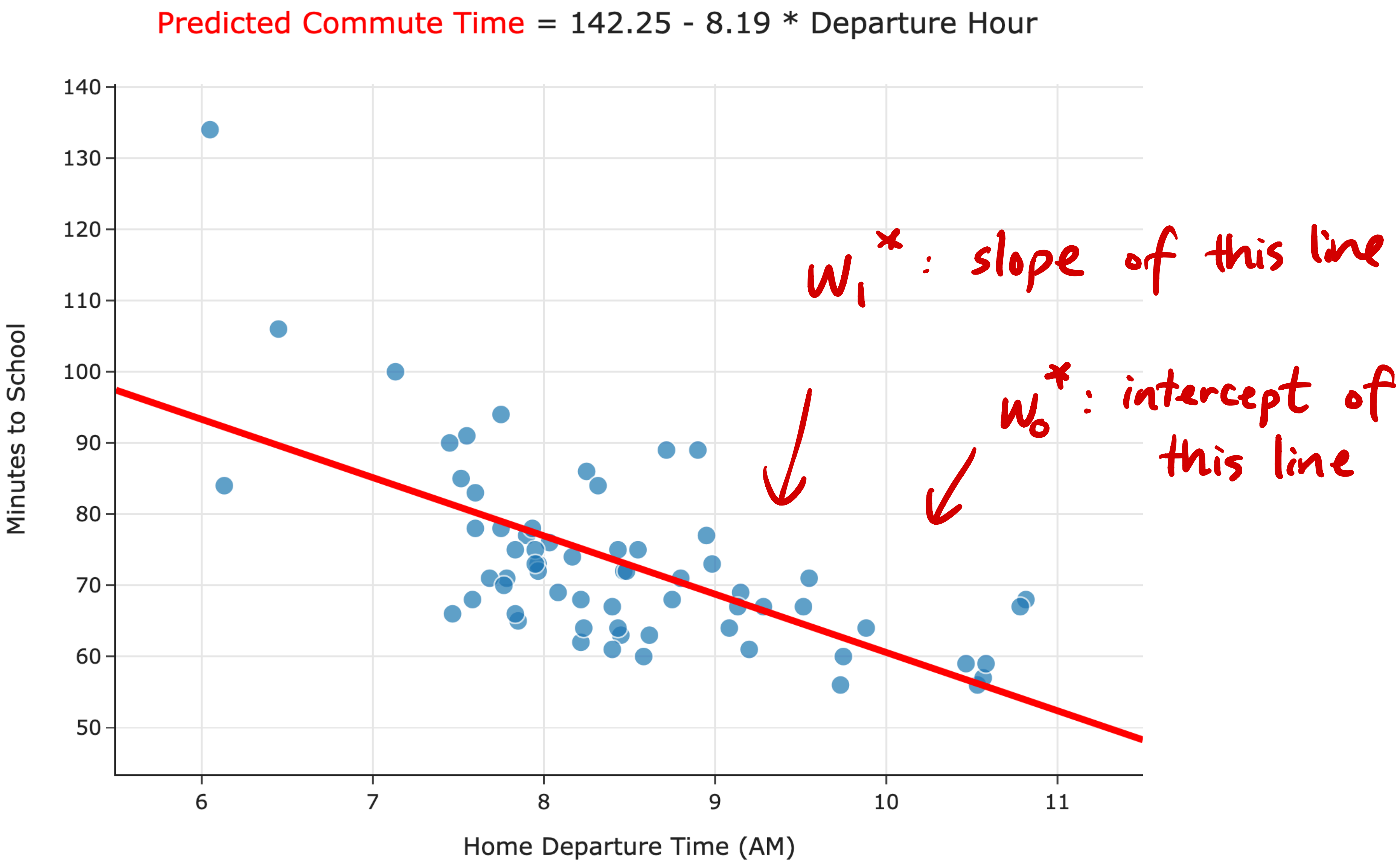
Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

Recap: Friends of simple linear regression


Predicted Commute Time = 142.25 - 8.19 * Departure Hour
Minutes to School
Home Departure Time (AM)
140
130
120
110
100
90
80
70
60
50
6
7
8
9
10
11
w_1^* : slope of this line
w_0^* : intercept of this line

Also equivalent to w_1^*: $\dfrac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \dfrac{\sum_{i=1}^{n}(y_i-\bar{y})x_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$

Simple linear regression

- Model: $H(x) = w_0 + w_1 x$.
- Loss function: squared loss, i.e. $L_{\text{sq}}(y_i, H(x_i)) = (y_i - H(x_i))^2$.
- Average loss, i.e. empirical risk:

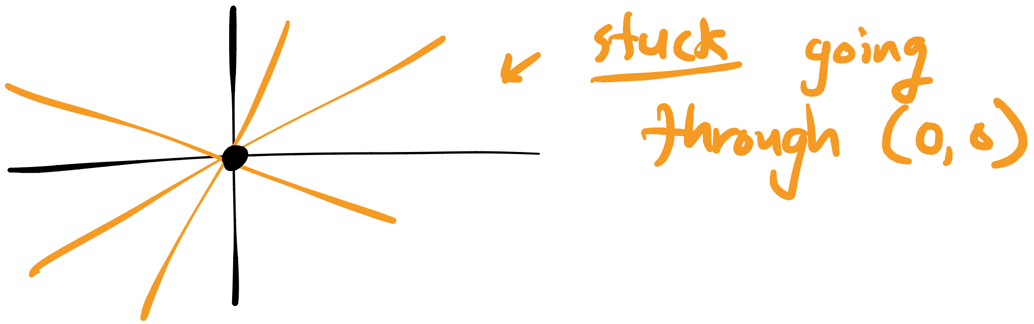
$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n}\sum_{i=1}^{n}(y_i - (w_0 + w_1 x_i))^2$$

- Optimal model parameters, found by minimizing empirical risk:

optimal slope ↓ | optimal intercept ↓

$$w_1^* = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = r\frac{\sigma_y}{\sigma_x} \qquad w_0^* = \bar{y} - w_1^*\bar{x}$$

Friends of simple linear regression



- Suppose we use squared loss throughout.
- If our model is $H(x) = w_1 x$, it is a **line that is forced through the origin,** $(0, 0)$.

$$w_1^* = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}$$

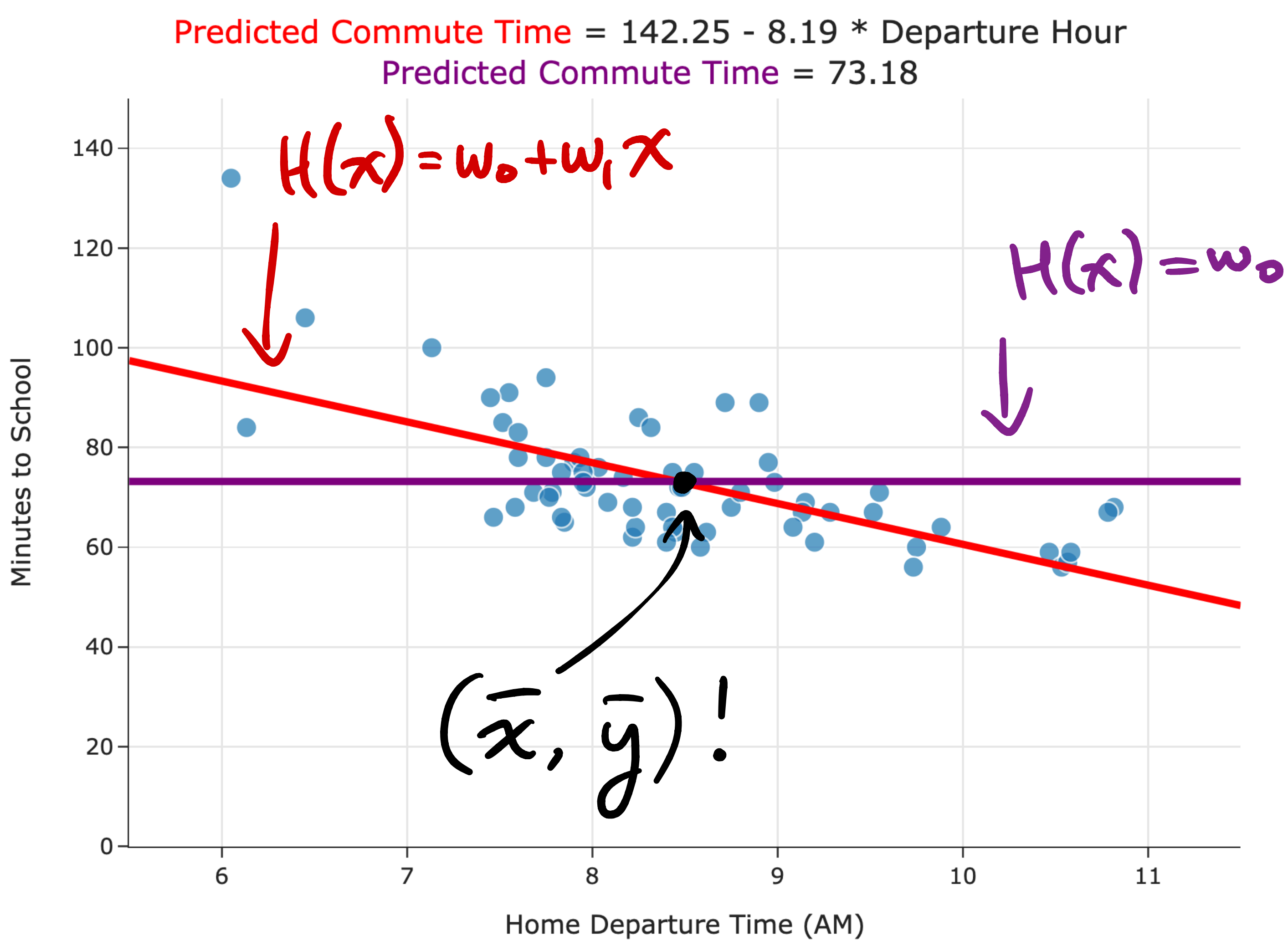
- If our model is $H(x) = w_0$, it is a **line that is forced to have a slope of** 0**, i.e. a horizontal line**. This is the same as the constant model from before.

$$w_0^* = \text{Mean}(y_1, y_2, \ldots, y_n)$$

- **Key idea**: w_0^* above is **not** necessarily equal to w_0^* for the simple linear regression model!

MSE(SLR) ≤ MSE(constant)

Comparing mean squared errors



$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(y_i - H(x_i))^2$$

- The MSE of the best simple linear regression model is ≈ 97.
- The MSE of the best constant model is ≈ 167. (variance)
- The simple linear regression model is a more flexible version of the constant model.

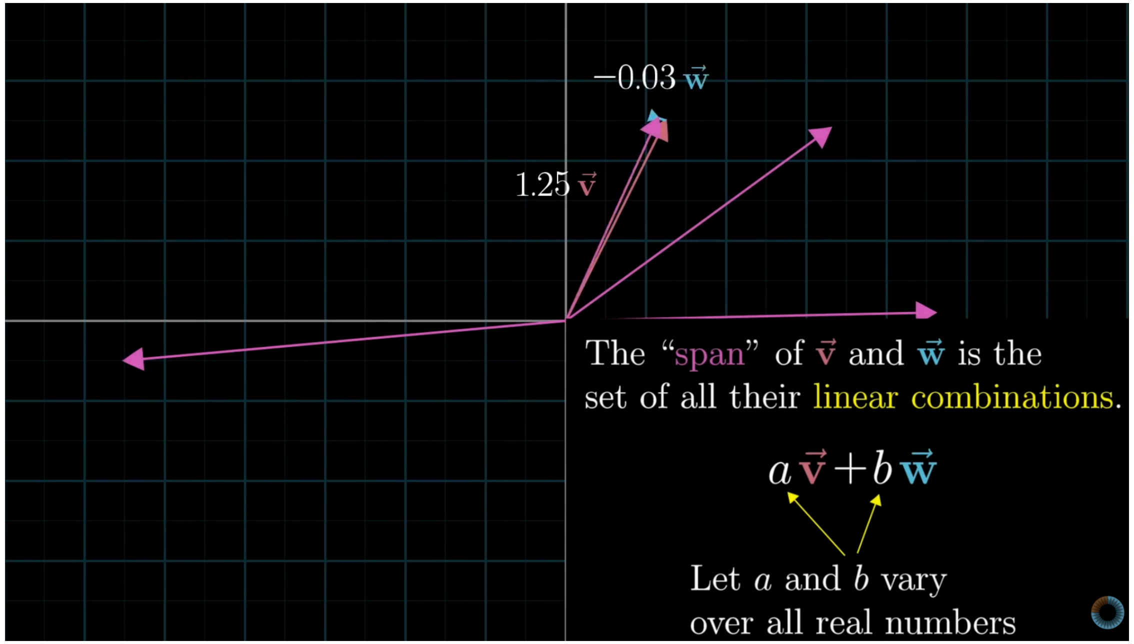
Dot products

Wait... why do we need linear algebra?

- Soon, we'll want to make predictions using more than one feature.
  - Example: Predicting commute times using departure hour and temperature.
- Thinking about linear regression in terms of **matrices and vectors** will allow us to find hypothesis functions that:
  - Use multiple features (input variables).
  - Are non-linear, e.g. $H(x) = w_0 + w_1 x + w_2 x^2$.
- Before we dive in, let's review.

Spans of vectors

- One of the most important ideas you'll need to remember from linear algebra is the concept of the **span** of one or more vectors.
- To jump start our review of linear algebra, let's start by watching 🎥 this video by 3blue1brown.



Warning ⚠️

- We're **not** going to cover every single detail from your linear algebra course.
- There will be facts that you're expected to remember that we won't explicitly say.
  - For example, if A and B are two matrices, then $AB \neq BA$.
  - This is the kind of fact that we will only mention explicitly if it's directly relevant to what we're studying.
  - But you still need to know it, and it may come up in homework questions.
- We **will** review the topics that you really need to know well.

in Overleaf: \mathbb{R}

$\mathbb{R}$: real numbers

n: there are n real numbers in our vector.

Vectors

- A **vector** in $\mathbb{R}^n$ is an **ordered collection of n numbers**.
- We use lower-case letters with an arrow on top to represent vectors, and we usually write vectors as **columns**.

$$\vec{v} = \begin{bmatrix} 8 \\ 3 \\ -2 \\ 5 \end{bmatrix}$$

- Another way of writing the above vector is $\vec{v} = [8, 3, -2, 5]^T$.

transpose

- Since $\vec{v}$ has four **components**, we say $\vec{v} \in \mathbb{R}^4$.

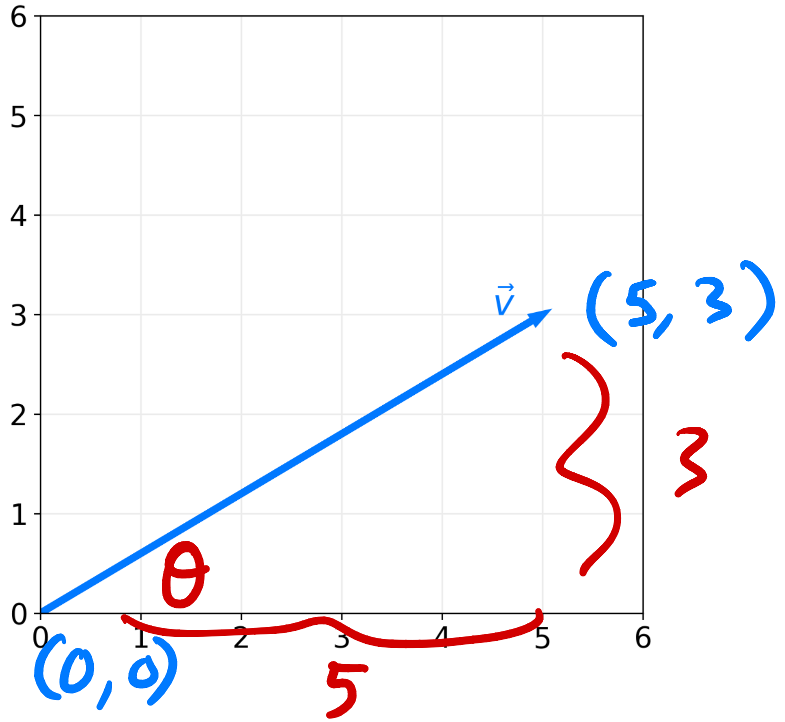
"elements"

"in"

$$\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\|\vec{v}\| = \sqrt{5^2 + 3^2} = \sqrt{34}$$

The geometric interpretation of a vector

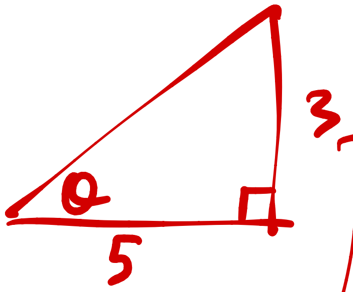


- A vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is an arrow to the point $(v_1, v_2, \ldots, v_n)$ from the origin.
- The **length**, or L_2 **norm**, of $\vec{v}$ is:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

multi-dimensional Pythagorean theorem

- A vector is sometimes described as an object with a **magnitude/length** and **direction**.



$$\tan\theta = \frac{3}{5}$$

$$\theta = \tan^{-1}\left(\frac{3}{5}\right)$$

both have the same number of elements
⇒ the same dimension

Dot product: coordinate definition

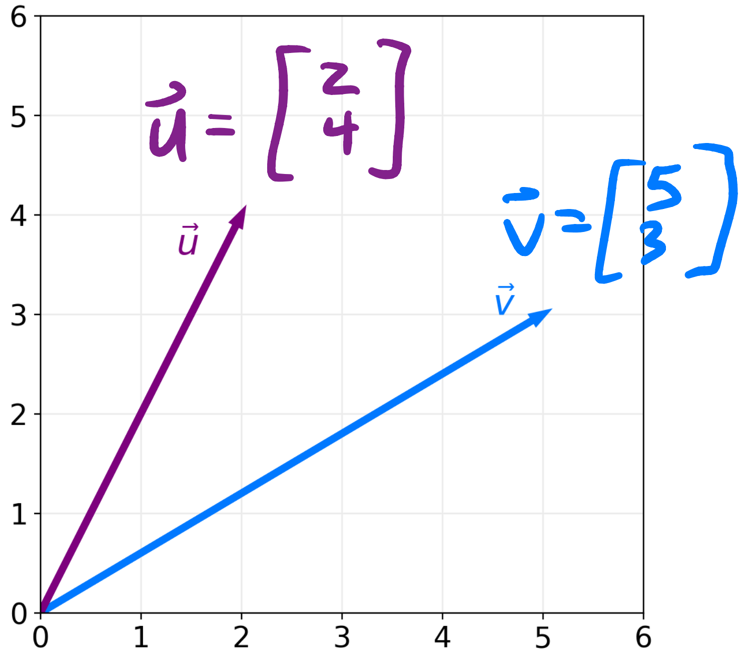
- The **dot product** of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is written as:

$$\vec{u} \cdot \vec{v} = \vec{u}^\mathsf{T} \vec{v}$$

- The computational definition of the dot product:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The result is a **scalar**, i.e. a single number.



$\vec{u} \cdot \vec{v} = (2)(5) + (4)(3) = 10 + 12 = \boxed{22}$ ← scalar! just one number!!!

$\vec{u}^\mathsf{T} \vec{v} = \begin{bmatrix} 2 & 4 \end{bmatrix}_{1\times 2} \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{2\times 1} = 22$

no dot!

match!

Question 🤔

Answer at q.dsc40a.com

Which of these is another expression for the length of $\vec{v}$?

- A. $\vec{v} \cdot \vec{v}$
- B. $\sqrt{\vec{v}^2}$
- C. $\sqrt{\vec{v} \cdot \vec{v}}$
- D. $\vec{v}^2$
- E. More than one of the above.

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$\vec{v}^2$ is undefined!

$\vec{v}_{n \times 1} \; \vec{v}_{n \times 1}$

inner dimensions must match! but, $1 \neq n$.

$\sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \|\vec{v}\|$

$$\sqrt{20}\sqrt{34}\cos\theta = 22 \Rightarrow \cos\theta = \frac{22}{\sqrt{20}\sqrt{34}}$$

$$\cos\theta = \frac{22}{2\sqrt{5}\sqrt{34}} = \frac{11}{\sqrt{5}\sqrt{34}}$$

Dot product: geometric definition

- The computational definition of the dot product:

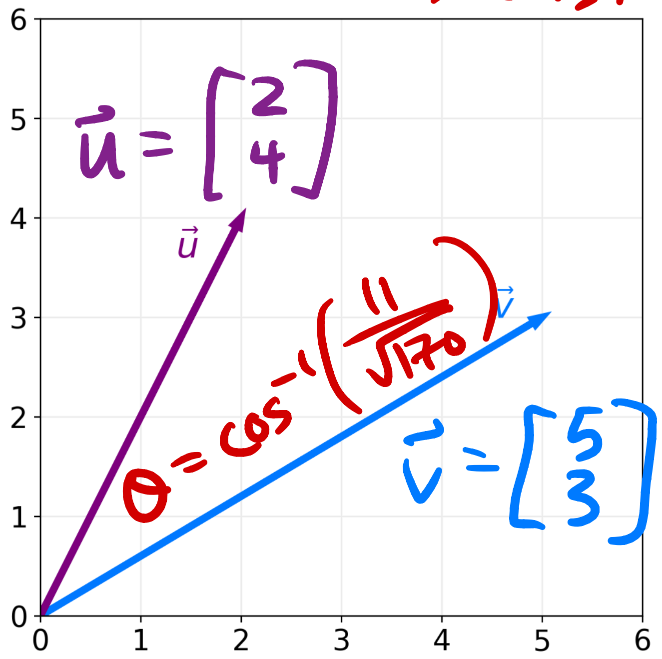
$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The geometric definition of the dot product:

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

- The two definitions are equivalent! This equivalence allows us to find the angle θ between two vectors.

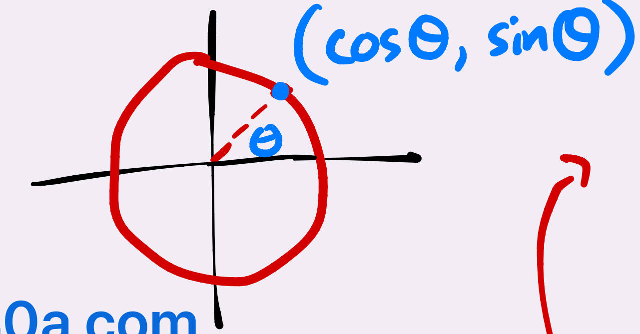


equal!

$$\vec{u}\cdot\vec{v} = (2)(5) + (4)(3) = 22$$

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta = \sqrt{2^2+4^2}\,\sqrt{5^2+3^2}\cos\theta = \sqrt{20}\sqrt{34}\cos\theta$$

Question 🤔



Answer at q.dsc40a.com

What is the value of θ in the plot to the right?

$\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

① $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$= 4(-2) + 1(-4) = -12$

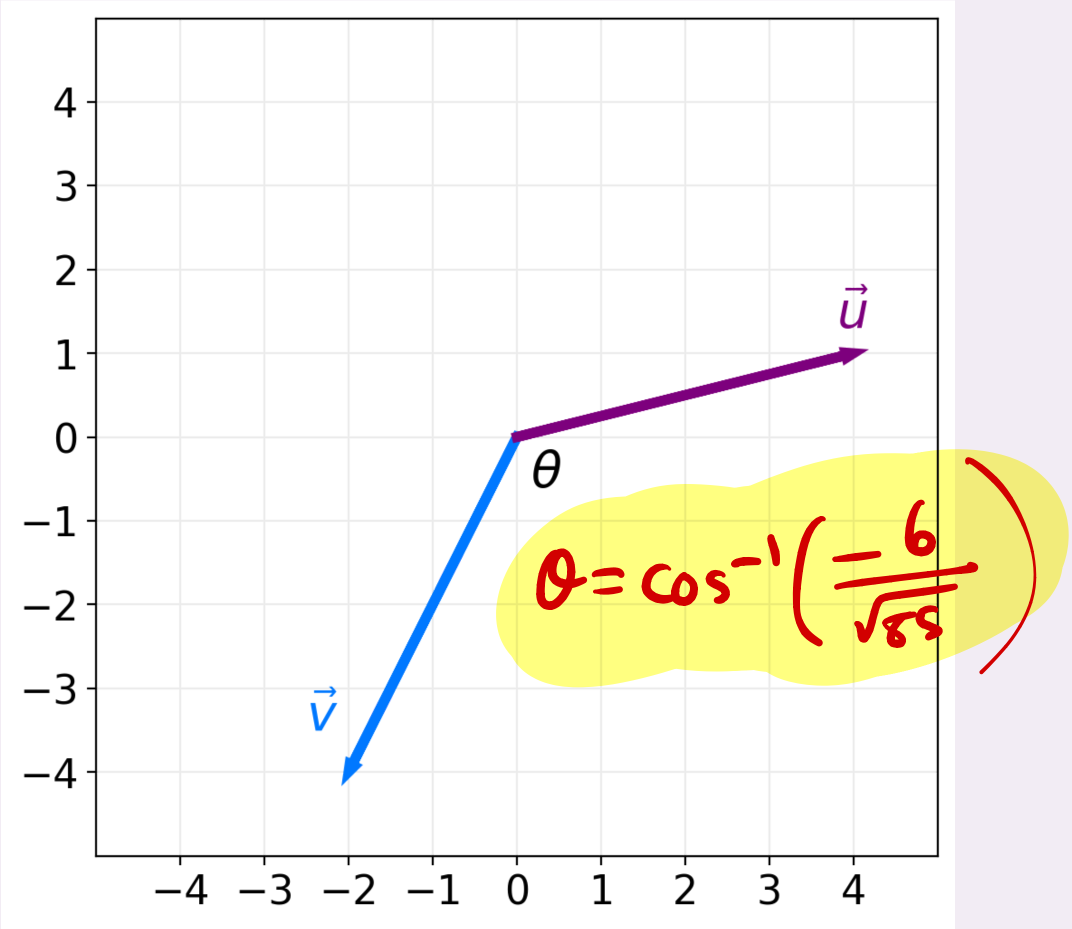
② $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$= \sqrt{4^2 + 1^2} \sqrt{(-2)^2 + (-4)^2} \cos\theta$

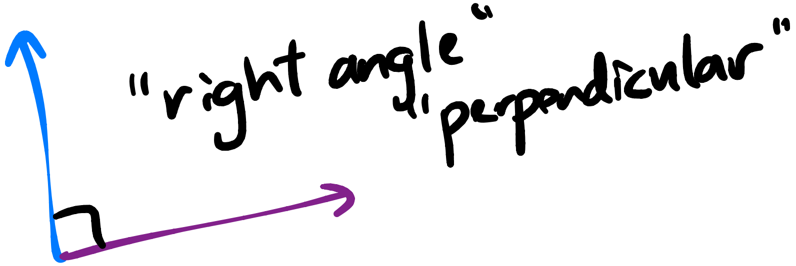
$= \sqrt{17} \sqrt{20} \cos\theta$

$\sqrt{17}\sqrt{20} \cos\theta = -12$

$\Rightarrow \cos\theta = \frac{-12}{\sqrt{17}\sqrt{20}} = \frac{-6}{\sqrt{17}\sqrt{5}} = \frac{-6}{\sqrt{85}}$



Orthogonal vectors



- Recall: $\cos 90^{\circ} = 0$.
- Since $\vec{u} \cdot \vec{v} = ||\vec{u}||\,||\vec{v}|| \cos\theta$, if the angle between two vectors is 90°, their dot product is $||\vec{u}||\,||\vec{v}|| \cos 90^{\circ} = 0$.
- If the angle between two vectors is 90°, we say they are perpendicular, or more generally, **orthogonal**.
- **Key idea**:

$$\boxed{\text{two vectors are } \textbf{orthogonal} \iff \vec{u} \cdot \vec{v} = 0}$$

"if and only if"
bidirectional statement

Exercise

Find a non-zero vector in $\mathbb{R}^3$ orthogonal to:

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -8 \end{bmatrix}$$

→ could find solutions to

$2u_1 + 5u_2 - 8u_3 = 0$

Infinitely many possibilities!

$\begin{bmatrix} 2 \\ 12 \\ 8 \end{bmatrix}$: $(2)(2) + (12)(5) + (8)(-8)$

$= 4 + 60 - 64$

$= 0$

$\begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix}$: $(0)(2) + (8)(5) + (5)(-8)$

$= 40 - 40$

$= 0$

Spans and projections

Adding and scaling vectors

$\vec{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow -2\vec{u} = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$

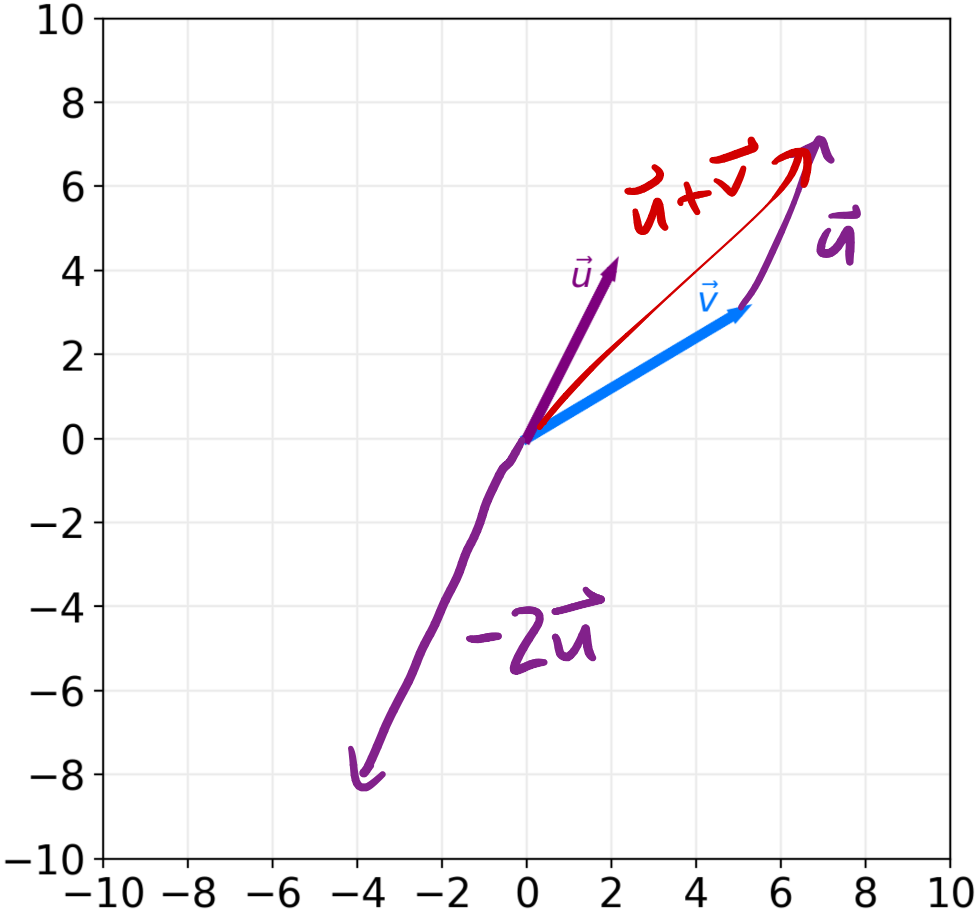
- The sum of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the **element-wise sum** of their components:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

← also a vector!

- If c is a scalar, then:

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$



d vectors,
each has n components

Linear combinations

- Let $\vec{v}_1, \vec{v}_2, ..., \vec{v}_d$ all be vectors in $\mathbb{R}^n$.
- A **linear combination** of $\vec{v}_1, \vec{v}_2, ..., \vec{v}_d$ is any vector of the form:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d$$

where $a_1, a_2, ..., a_d$ are all scalars.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

Examples

$$2\vec{v}_1 + \vec{v}_2 + \frac{1}{9}\vec{v}_3 = \begin{bmatrix} \_ \\ \_ \end{bmatrix}$$ a vector in $\mathbb{R}^2$!

$$0\vec{v}_1 + \vec{v}_2 - \vec{v}_3$$

⋮

Span

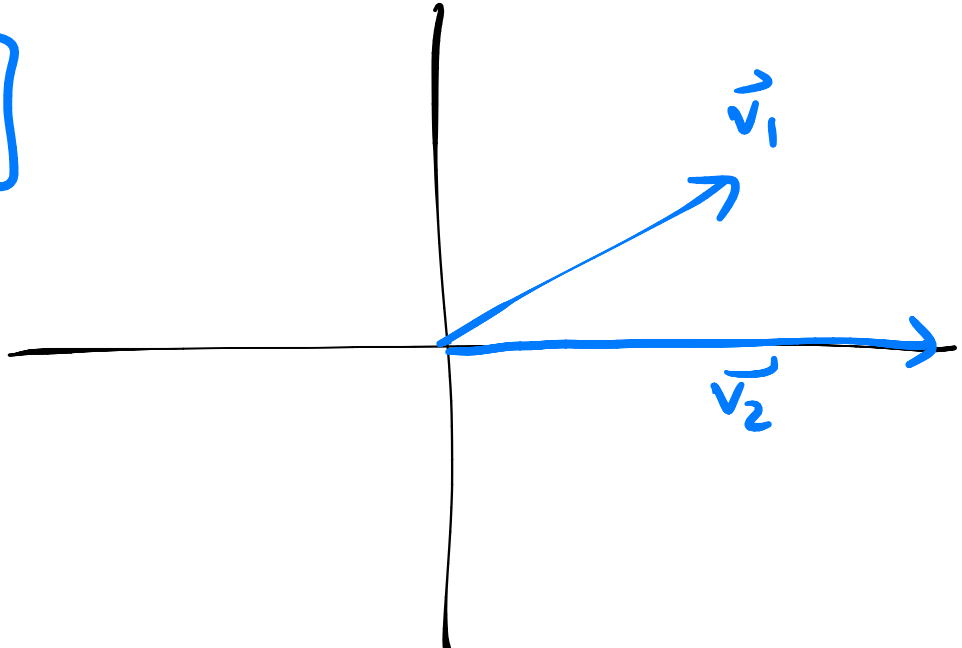
- Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ all be vectors in $\mathbb{R}^n$.
- The **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ is the set of all vectors that can be created using linear combinations of those vectors.
- Formal definition:

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d : a_1, a_2, \ldots, a_d \in \mathbb{R}\}$$

Example

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$\vec{v}_1$ and $\vec{v}_2$ span all of $\mathbb{R}^2$!



We can! $\vec{v}_1$ and $\vec{v}_2$ aren't scalar multiples of each other: they point in diff. directions

Exercise

Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and let $\vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Is $\vec{y} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2)$?

If so, write $\vec{y}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

$$w_1 \vec{v}_1 + w_2 \vec{v}_2 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2w_1 \\ -3w_1 \end{bmatrix} + \begin{bmatrix} -w_2 \\ 4w_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2w_1 - w_2 &= 9 \\ -3w_1 + 4w_2 &= 1 \end{aligned} \longrightarrow \text{solve for } w_1, w_2.$$

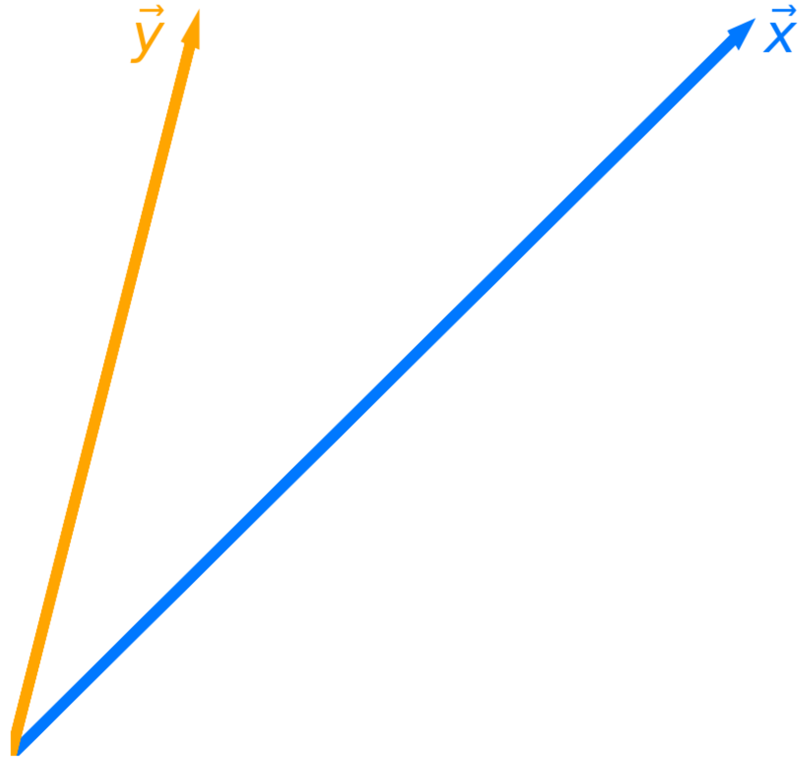
Projecting onto a single vector

- Let $\vec{x}$ and $\vec{y}$ be two vectors in $\mathbb{R}^n$.
- The span of $\vec{x}$ is the set of all vectors of the form:

$$w\vec{x}$$

  where $w \in \mathbb{R}$ is a scalar.

- **Question**: What vector in $\text{span}(\vec{x})$ is closest to $\vec{y}$?
- The vector in $\text{span}(\vec{x})$ that is closest to $\vec{y}$ is the **projection of $\vec{y}$ onto** $\text{span}(\vec{x})$.



Projection error

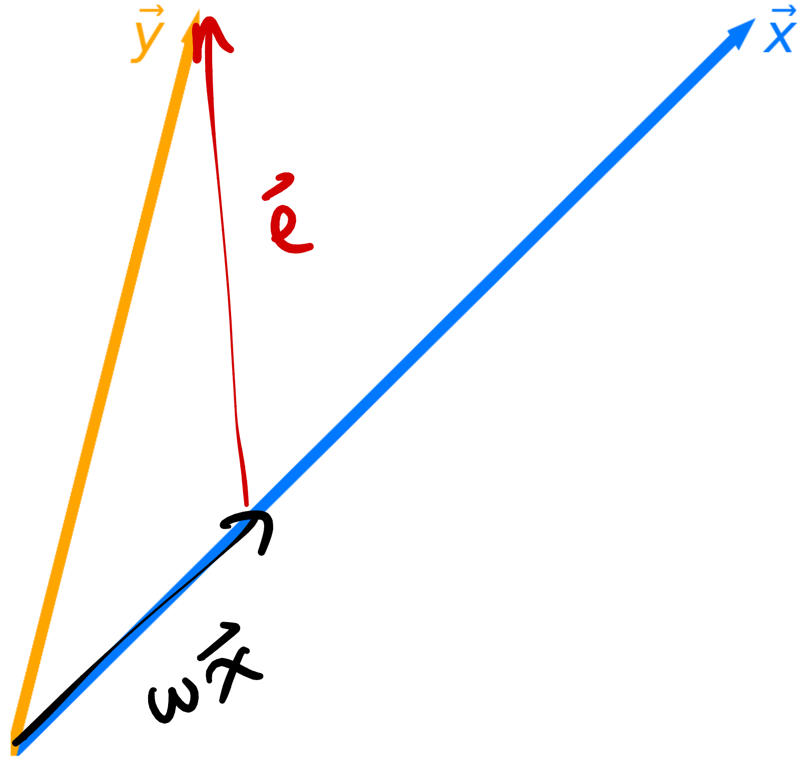
- Let $\vec{e} = \vec{y} - w\vec{x}$ be the **projection error**: that is, the vector that connects $\vec{y}$ to $\text{span}(\vec{x})$.
- **Goal**: **Find the w that makes $\vec{e}$ as short as possible.**
  - That is, minimize:

$$||\vec{e}||$$

  - Equivalently, minimize:

$$||\vec{y} - w\vec{x}||$$

- **Idea**: To make $\vec{e}$ has short as possible, it should be **orthogonal to** $w\vec{x}$.



Minimizing projection error

- **Goal**: **Find the w that makes $\vec{e} = \vec{y} - w\vec{x}$ as short as possible**.
- **Idea**: To make $\vec{e}$ as short as possible, it should be **orthogonal to $w\vec{x}$**.
- Can we prove that making $\vec{e}$ orthogonal to $w\vec{x}$ minimizes $||\vec{e}||$?

Minimizing projection error

- **Goal**: **Find the w that makes $\vec{e} = \vec{y} - w\vec{x}$ as short as possible**.
- Now we know that to minimize $||\vec{e}||$, $\vec{e}$ must be orthogonal to $w\vec{x}$.
- Given this fact, how can we solve for w?

Orthogonal projection

- **Question**: What vector in $\text{span}(\vec{x})$ is closest to $\vec{y}$?
- **Answer**: It is the vector $w^* \vec{x}$, where:

$$w^* = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}$$

- Note that w^* is the solution to a minimization problem, specifically, this one:

$$\text{error}(w) = \|\vec{e}\| = \|\vec{y} - w\vec{x}\|$$

- We call $w^* \vec{x}$ the **orthogonal projection of** $\vec{y}$ **onto** $\text{span}(\vec{x})$.
    - Think of $w^* \vec{x}$ as the "shadow" of $\vec{y}$.

Exercise

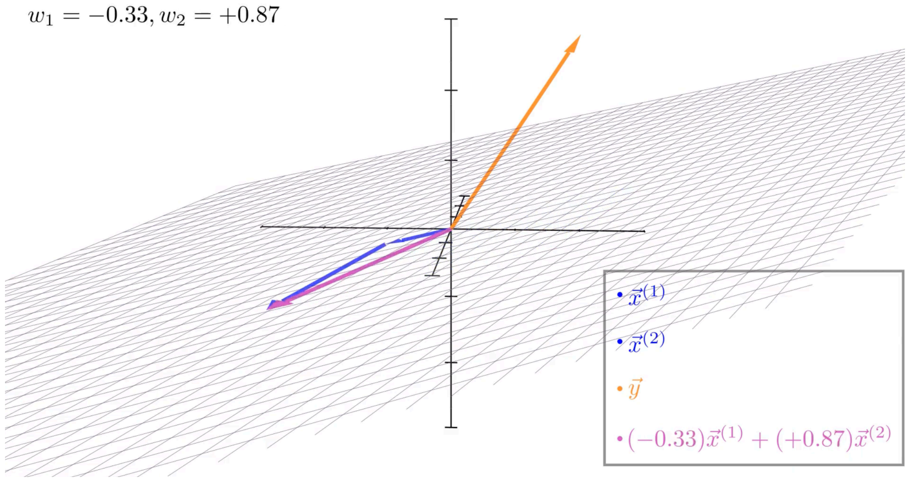
Let $\vec{a} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -1 \\ 9 \end{bmatrix}$.

What is the orthogonal projection of $\vec{a}$ onto $\text{span}(\vec{b})$?

Your answer should be of the form $w^* \vec{b}$, where w^* is a scalar.

Moving to multiple dimensions

- Let's now consider three vectors, $\vec{y}$, $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$, all in $\mathbb{R}^n$.
- **Question**: What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to $\vec{y}$?
  - Vectors in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ are of the form $w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}$, where $w_1, w_2 \in \mathbb{R}$ are scalars.
- Before trying to answer, let's watch 🎥 **this animation that Jack, one of our tutors, made**.



Minimizing projection error in multiple dimensions

- **Question**: What vector in $\text{span}(\vec{x}^{(1)}, \vec{x}^{(2)})$ is closest to $\vec{y}$?
  - That is, what vector minimizes $\|\vec{e}\|$, where:

$$\vec{e} = \vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)}$$

- **Answer**: It's the vector such that $w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)}$ is **orthogonal** to $\vec{e}$.
- **Issue**: Solving for w_1 and w_2 in the following equation is difficult:

$$\left(w_1 \vec{x}^{(1)} + w_2 \vec{x}^{(2)}\right) \cdot \underbrace{\left(\vec{y} - w_1 \vec{x}^{(1)} - w_2 \vec{x}^{(2)}\right)}_{\vec{e}} = 0$$

What's next?

- It's hard for us to solve for w_1 and w_2 in:

$$\left(w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}\right) \cdot \underbrace{\left(\vec{y} - w_1\vec{x}^{(1)} - w_2\vec{x}^{(2)}\right)}_{\vec{e}} = 0$$

- **Solution**: Combine $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ into a single **matrix**, X, and express $w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)}$ as a **matrix-vector multiplication**, Xw.

- **Next time**: Formulate linear regression in terms of matrices and vectors!