Lecture 17 – Sequences, Permutations, and Combinations

DSC 40A, Spring 2023

Announcements

- Homework 5 is due **Tuesday at 11:59pm**.
- **Important:** We've posted **many** probability resources on the resources tab of the course website. These will no doubt come in handy.
  - No more DSC 40A-specific readings, though the Probability Roadmap was written specifically for students of this course.

Agenda

- Sequences, permutations, and combinations.
- Lots of examples.

Sequences, permutations, and combinations

Motivation

- Many problems in probability involve counting.
  - Suppose I flip a fair coin 100 times. What's the probability I see 34 heads?
  - Suppose I draw 3 cards from a 52 card deck. What's the probability they all are all from the same suit?
- In order to solve such problems, we first need to learn how to count.
- The area of math that deals with counting is called **combinatorics**.

Selecting elements (i.e. sampling)

- Many experiments involve choosing k elements randomly from a group of n possible elements. This group is called a **population**.
  - If drawing cards from a deck, the population is the deck of all cards.
  - If selecting people from DSC 40A, the population is everyone in DSC 40A.
- Two decisions:
  - Do we select elements with or without **replacement**?
  - Does the **order** in which things are selected matter?

Sequences

- A **sequence** of length k is obtained by selecting k elements from a group of n possible elements **with replacement**, such that **order matters**.

- **Example:** Draw a card (from a standard 52-card deck), put it back in the deck, and repeat 4 times.

Q♢, Q♢, Q♢, Q♢ is possible

$$52 \cdot 52 \cdot 52 \cdot 52 = (52)^4$$

- **Example:** A UCSD PID starts with "A" then has 8 digits. How many UCSD PIDs are possible?

$$(10)^8$$

why multiply?

$3 * 2 = 6$ outfits

Sequences

In general, the number of ways to select k elements from a group of n possible elements such that **repetition is allowed** and **order matters** is $\boxed{n^k.}$

(Note: We mentioned this fact in the lecture on clustering!)

$52\cdot51\cdot50\cdot49 = \dfrac{52\cdot51\cdot50\cdot49\cdot\cancel{48}\cdot\cancel{47}\cdots}{\cancel{48}\cdot\cancel{47}\cdots}$

Permutations

- A **permutation** is obtained by selecting *k* elements from a group of *n* possible elements **without replacement**, such that **order matters**.

$n=52$
$k=4$

- **Example:** Draw 4 cards (without replacement) from a standard 52-card deck. $P(52,4)$

no longer possible to get Q◇, Q◇, Q◇, Q◇

$$n \rightarrow 52\cdot51\cdot50\cdot49 \leftarrow n-k+1$$

- **Example:** How many ways are there to select a president, vice president, and secretary from a group of 8 people?

pres vp sec.

$$8\cdot7\cdot6$$

n $\quad$ $n-k+1$

$n=8$
$k=3$

$P(8,3)$

Permutations

- In general, the number of ways to select k elements from a group of n possible elements such that **repetition is not allowed** and **order matters** is

$$P(n,k) = (n)(n-1)\ldots(n-k+1)$$

- To simplify: recall that the definition of $n!$ is

$$n! = (n)(n-1)\ldots(2)(1)$$

- Given this, we can write

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$\frac{n\cdot(n-1)\ldots\cdot(n-k+1)\cancel{(n-k)(n-k-1)\ldots(1)}}{\cancel{(n-k)(n-k-1)\ldots(1)}} = \frac{n!}{(n-k)!}$$

order matters
no repetitions > permutation
$n=7, k=3$

Discussion Question

UCSD has 7 colleges. How many ways can I rank my top 3 choices?

a) 21
b) 210
c) 343
d) 2187
e) None of the above.

$$P(7,3) = \frac{n!}{(n-k)!} = \frac{7!}{(7-3)!} = \frac{7!}{4!}$$

$$= \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}} = \boxed{7\cdot 6\cdot 5}$$

ex.) Sixth, Warren, ERC
7, 6, 5
$\stackrel{?}{=}$ Warren, ERC, Sixth > order matters, not considered same

Special case of permutations

rearrangements of n objects is $n!$

$n = k$

- Suppose we have n people. The total number of ways I can rearrange these n people in a line is

$P(n, n) = n!$

people labeled a, b, c, d, e

$$\underset{5}{\underline{e}} \cdot \underset{4}{\underline{a}} \cdot \underset{3}{\underline{c}} \cdot \underset{2}{\underline{b}} \cdot \underset{1}{\underline{d}}$$

- This is consistent with the formula

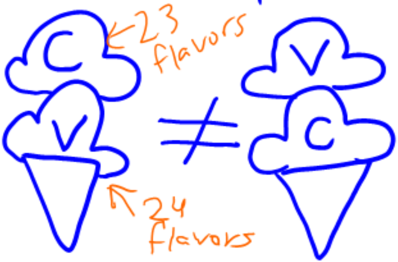
$$P(n, n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

$$\boxed{0! = 1}$$

Combinations - which elements are included

- A **combination** is a set of k items selected from a group of n possible elements **without replacement**, such that **order does not matter**. → sets - don't have repeats
- **Example:** There are 24 ice cream flavors. How many ways can you pick two flavors?

how many flavor combos? → $VC = CV$ $\quad 24 \cdot 23/2$



← how many cones possible if order does matter?

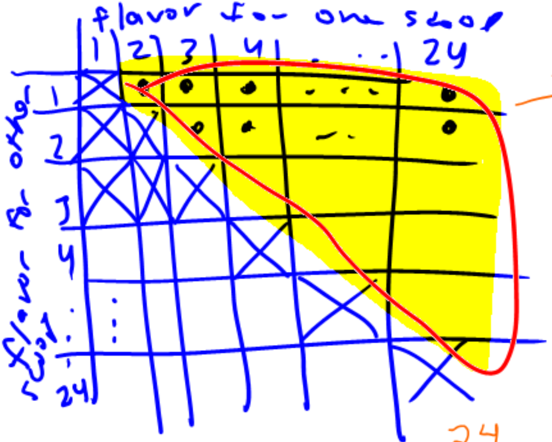
$$P(24,2) = \frac{24!}{22!} = 24 \cdot 23$$

$VC \neq CV$

above diagonal = 2-flavor cones

diagonal = duplicate flavor cones

below diagonal = also corresponds to 2 flavor cones, but these are already counted by above diag.



how many flavor combos? count across rows!

$$\frac{23}{2}\cdot 24 = 11.5 * 24 = 11*24+12$$

$$23+22+21+\ldots+3+2+1$$

24

24

24

each pair adds to 24, but 23 terms. 11.5 pairs

3-scoop cones

24 flavor options

how many 3-flavor combos?

- can't duplicate flavors
- no order

$$\frac{24\cdot 23\cdot 22}{6} = \frac{P(24,3)}{3!}$$

simpler problem: order does matter

$24\cdot 23\cdot 22 = P(24,3)$

← 22

← 23

← 24

adjust for order not mattering

$3! = 6$ orderings of any set of 3 flavors

From permutations to combinations

- There is a close connection between:
  - the number of permutations of k elements selected from a group of n, and
  - the number of combinations of k elements selected from a group of n

$$\text{\# combinations} = \frac{\text{\# permutations}}{\text{\# orderings of } k \text{ items}} = \frac{P(n, k)}{k!}$$

- Since # permutations = $\frac{n!}{(n-k)!}$ and # orderings of k items = $k!$, we have

not a fraction

$$C(n, k) = \binom{n}{k} = \boxed{\frac{n!}{(n-k)!k!}} = \frac{\frac{n!}{(n-k)!}}{k!}$$

Combinations (order doesn't matter)

Combinations

In general, the number of ways to select k elements from a group of n elements such that **repetition is not allowed** and **order does not matter** is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

The symbol $\binom{n}{k}$ is pronounced "n choose k", and is also known as the **binomial coefficient**.

Example: committees

- How many ways are there to select a president, vice president, and secretary from a group of 8 people?

$P(8,3) = 8 \cdot 7 \cdot 6$

- How many ways are there to select a committee of 3 people from a group of 8 people?

$C(8,3) = \frac{8 \cdot 7 \cdot 6}{3!}$

- If you're ever confused about the difference between permutations and combinations, **come back to this example**.

Discussion Question

A domino consists of two faces, each with anywhere between 0 and 6 dots. A set of dominoes consists of every possible combination of dots on each face.
How many dominoes are in the set of dominoes?

a) $\binom{7}{2}$
b) $\binom{7}{1} + \binom{7}{2}$
c) $P(7, 2)$
d) $\frac{P(7,2)}{P(7,1)} 7!$

More examples

Counting and probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $P(A) = \frac{|A|}{|S|}$.
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip:** Before starting a probability problem, always think about what the sample space S is!

Selecting students — overview

We're going to start by answering the same question using several different techniques.

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 1: using permutations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 2: using permutations and the complement)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 3: using combinations)

Question 1, Part 1 (Denominator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals could you draw?

Selecting students (Method 3: using combinations)

Question 1, Part 2 (Numerator): If you draw a sample of size 5 at random without replacement from a population of size 20, how many different **sets** of individuals include Avi?

Selecting students (Method 3: using combinations)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students (Method 4: “the easy way”)

Question 1: There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

With vs. without replacement

Discussion Question

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

a) Equal to
b) Greater than
c) Less than

Art supplies

Question 2, Part 1: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies?

Art supplies

Question 2, Part 2: We have 12 art supplies: 5 markers and 7 crayons. In how many ways can we select 4 art supplies such that we have...

1. 2 markers and 2 crayons?
2. 3 markers and 1 crayon?
3. At least 2 markers?

Art supplies

Question 2, Part 3: We have 12 art supplies: 5 markers and 7 crayons. We randomly select 4 art supplies. What's the probability that we selected at least 2 markers?

Fair coin

Question 3: Suppose we flip a fair coin 10 times. What is the probability that we see an equal number of heads and tails?

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Unfair coin

Question 4: Suppose we flip a coin **that is not fair**, but instead has $P(\text{heads}) = \frac{1}{3}$, 10 times. Assume that each flip is independent.

1. What is the probability that we see the specific sequence THTTHTHHTH?
2. What is the probability that we see an equal number of heads and tails?

Summary

Summary

- A **sequence** is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters.
  - Number of sequences: n^k.
- A **permutation** is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters.
  - Number of permutations: $P(n,k) = \frac{n!}{(n-k)!}$.
- A **combination** is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter.
  - Number of combinations: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.