

Lecture 5 – Gradient Descent



DSC 40A, Spring 2023

Announcements

- ▶ Discussion is tonight at 7pm or 8pm in FAH 1101.
 - ▶ Come to work on Groupwork 2, which is due **tonight at 11:59pm.**
 - ▶ Please attend the section you are enrolled in.
- ▶ Homework 1 deadline extended to **Thursday at 11:59pm.**
 - ▶ This is a one-time bonus for the first homework. I will review your submission today and let you know if the amount of explanation provided seems insufficient.
 - ▶ No submissions after Thursday. Nobody will use a slip day on Homework 1.
- ▶ Homework 2 is released, due **Tuesday at 11:59pm.**

Agenda

- ▶ Brief recap of Lecture 4.
- ▶ Gradient descent fundamentals.

Empirical risk minimization

The recipe

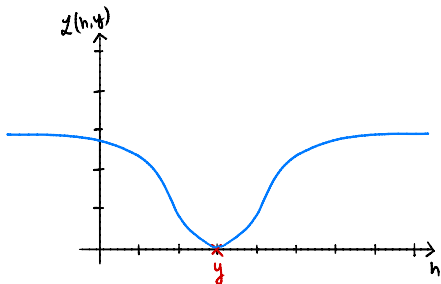
Suppose we're given a dataset, y_1, y_2, \dots, y_n and want to determine the best future prediction h^* .

1. Choose a loss function $L(h, y)$ that measures how far our prediction h is from the “right answer” y .
 - ▶ Absolute loss, $L_{abs}(h, y) = |y - h|$.
 - ▶ Squared loss, $L_{sq}(h, y) = (y - h)^2$.
2. Find h^* by minimizing the average of our chosen loss function over the entire dataset.
 - ▶ “Empirical risk” is just another name for average loss.

$$R(h) = \frac{1}{n} \sum_{i=1}^n L(h, y_i)$$

A very insensitive loss

- ▶ Last time, we introduced a new loss function, L_{ucsd} , with the property that it (roughly) penalizes all bad predictions the same.
 - ▶ A prediction that is off by 50 has approximately the same loss as a prediction that is off by 500.
 - ▶ The effect: L_{ucsd} is not as sensitive to outliers.



A very insensitive loss

- ▶ The formula for L_{ucsd} is as follows (no need to memorize):

$$L_{ucsd}(h, y) = 1 - e^{-(y-h)^2/\sigma^2}$$

- ▶ The shape (and formula) come from an upside-down bell curve.
- ▶ L_{ucsd} contains a **scale parameter**, σ .
 - ▶ Nothing to do with variance or standard deviation.
 - ▶ Accounts for the fact that different datasets have different thresholds for what counts as an outlier.
 - ▶ Like a knob that you get to turn – the larger σ is, the more sensitive L_{ucsd} is to outliers (and the more smooth R_{ucsd} is).

Minimizing R_{ucsd}

- ▶ The corresponding empirical risk, R_{ucsd} , is

$$R_{ucsd}(h) = \frac{1}{n} \sum_{i=1}^n [1 - e^{-(y_i-h)^2/\sigma^2}]$$

- ▶ R_{ucsd} is **differentiable**.
- ▶ To minimize: take derivative, set to zero, solve.

Step 1: Taking the derivative

$$\frac{dR_{ucsd}}{dh} = \frac{d}{dh} \left(\frac{1}{n} \sum_{i=1}^n [1 - e^{-(y_i-h)^2/\sigma^2}] \right)$$

Step 2: Setting to zero and solving

- ▶ We found:

$$\frac{d}{dh} R_{ucsd}(h) = \frac{2}{n\sigma^2} \sum_{i=1}^n (h - y_i) \cdot e^{-(h-y_i)^2/\sigma^2}$$

- ▶ Now we just set to zero and solve for h :

$$0 = \frac{2}{n\sigma^2} \sum_{i=1}^n (h - y_i) \cdot e^{-(h-y_i)^2/\sigma^2}$$

- ▶ We **can** calculate derivative, but we **can't** solve for h ; we're stuck again.

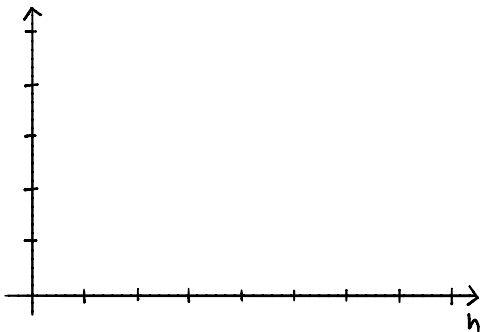
Gradient descent fundamentals

The general problem

- ▶ **Given:** a differentiable function $R(h)$.
- ▶ **Goal:** find the input h^* that minimizes $R(h)$.

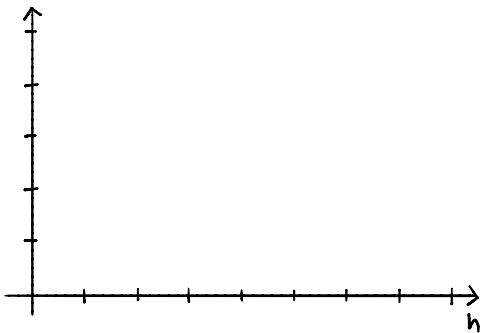
Meaning of the derivative

- ▶ We're trying to minimize a **differentiable** function $R(h)$. Is calculating the derivative helpful?
- ▶ $\frac{dR}{dh}(h)$ is a function; it gives the **slope** at h .



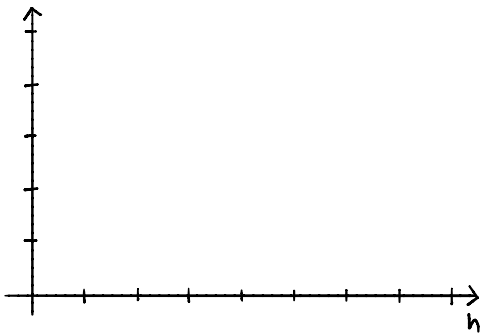
Key idea behind **gradient descent**

- ▶ If the slope of R at h is **positive** then moving to the **left** decreases the value of R .
- ▶ i.e., we should **decrease** h .



Key idea behind **gradient descent**

- ▶ If the slope of R at h is **negative** then moving to the **right** decreases the value of R .
- ▶ i.e., we should **increase** h .



Key idea behind **gradient descent**

- ▶ Pick a starting place, h_0 . Where do we go next?
- ▶ Slope at h_0 negative? Then increase h_0 .
- ▶ Slope at h_0 positive? Then decrease h_0 .

Key idea behind **gradient descent**

- ▶ Pick a starting place, h_0 . Where do we go next?
- ▶ Slope at h_0 negative? Then increase h_0 .
- ▶ Slope at h_0 positive? Then decrease h_0 .
- ▶ Something like this will work:

$$h_1 = h_0 - \frac{dR}{dh}(h_0)$$

Gradient Descent

- ▶ Pick α to be a positive number. It is the **learning rate**, also known as the **step size**.
- ▶ Pick a starting prediction, h_0 .
- ▶ On step i , perform update $h_i = h_{i-1} - \alpha \cdot \frac{dR}{dh}(h_{i-1})$
- ▶ Repeat until convergence (when h doesn't change much).

Gradient Descent

```
def gradient_descent(derivative, h, alpha, tol=1e-12):  
    """Minimize using gradient descent."""  
    while True:  
        h_next = h - alpha * derivative(h)  
        if abs(h_next - h) < tol:  
            break  
        h = h_next  
    return h
```

Note: it's called gradient descent because the gradient is the generalization of the derivative for multivariable functions.

Example: Minimizing mean squared error

- ▶ Recall the mean squared error and its derivative:

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^n (y_i - h)^2 \quad \frac{dR_{\text{sq}}}{dh}(h) = \frac{2}{n} \sum_{i=1}^n (h - y_i)$$

Discussion Question

Consider the dataset $-4, -2, 2, 4$. Pick $h_0 = 4$ and $\alpha = \frac{1}{4}$. Find h_1 .

- a) -1
- b) 0
- c) 1
- d) 2

Solution

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^n (y_i - h)^2 \quad \frac{dR_{\text{sq}}}{dh}(h) = \frac{2}{n} \sum_{i=1}^n (h - y_i)$$

Consider the dataset $-4, -2, 2, 4$. Pick $h_0 = 4$ and $\alpha = \frac{1}{4}$. Find h_1 .

Summary

- ▶ Gradient descent is a general tool used to minimize differentiable functions.
 - ▶ We will usually use it to minimize empirical risk, but it can minimize other functions, too.
- ▶ Gradient descent progressively updates our guess for h^* according to the update rule

$$h_i = h_{i-1} - \alpha \cdot \left(\frac{dR}{dh}(h_{i-1}) \right).$$

- ▶ **Next Time:** We'll demonstrate gradient descent in a Jupyter notebook. We'll learn when this procedure works well and when it doesn't.