DSC 40A - Homework 6

Due: Tuesday, November 16 at 11:59pm

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. Homeworks are due to Gradescope by 11:59pm on the due date. You can use a slip day to extend the deadline by 24 hours. Make sure to correctly assign pages to Gradescope when submitting.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain and justify** your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Homeworks should be written up and turned in by each student individually. You may talk to other students in the class about the problems and discuss solution strategies, but you should not share any written communication and you should not check answers with classmates. You can tell someone how to do a homework problem, but you cannot show them how to do it.

For each problem you submit, you should **cite your sources** by including a list of names of other students with whom you discussed the problem. Instructors do not need to be cited.

This homework will be graded out of 42 points. The point value of each problem or sub-problem is indicated by the number of avocados shown.

Throughout this homework, you are allowed to leave your answers in terms of factorials, the permutation formula $P(n,k)$, and the binomial coefficient $\binom{n}{k}$, unless otherwise stated.

Problem 1. Pass the Aux!

You're hosting a party and need to pick music, but you aren't up-to-date with the latest tracks on the charts. So, you ask 13 of your friends to each send you a single song to play at the party.

a) If each of your friends sends you a different song, how many different orders could you play the 13 songs in?

b) Now suppose that 3 of your 13 friends happen to send you the same song, "Industry Baby" by Lil Nas X and Jack Harlow. You told your friends that you would play each song as many times as it was requested; in other words, you have to play "Industry Baby" 3 times. Assume that your other 10 friends each sent you different songs.

How many different orders could you play the 13 songs in now?

Hint: There is a question in the groupwork that is very similar to parts (b) and (c) of this question.

c) Suppose that your other 10 friends catch on to the fact that you're playing "Industry Baby" 3 times, and decide to send you a bunch of repeated songs as well. Here are the 13 songs that you received from them, along with the number of times each song was sent:

- "Industry Baby" (Lil Nas X and Jack Harlow) 3 times
- "Need to Know" (Doja Cat) 3 times
- "brutal" (Olivia Rodrigo) 2 times
- "No Friends In The Industry" (Drake) 5 times

How many different orders could you play the 13 songs in now?

Problem 2. Pascal's Identity

Suppose I have 10 masks, each of which is a different color, and I want to create a combination of 5 of them. From lecture, we know that this can be done in $\binom{10}{5}$ ways. In this problem, we'll look at another way to arrive at this same result.

a) Let's consider my purple mask. How many ways can I select 5 masks from my group of 10 masks such that my purple mask is one of the 5 selected?

b) How many ways can I select 5 masks from my group of 10 masks such that my purple mask is **not** one of the 5 selected?

c) Using the results of the previous two parts, how many ways can I select 5 masks from my group of 10 masks?

Note: You must write your answer in terms of your results to parts (a) and (b); you will get no credit if you write $\binom{10}{5}$.

d) What you've just discovered is an application of Pascal's Identity, which states that

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Prove Pascal's identity.

Hint: Start with the left-hand side, use the definition of the binomial coefficient, and try and bring both terms to a common denominator. Another hint — what is $\frac{n}{n!}$?

Side note: Pascal's identity has a close connection to Pascal's triangle.

Problem 3. McDonald's Ice Cream Machines

Suppose that every time you go to McDonald's, there is a 5% chance that the ice cream machine is broken.

a) Fix an integer $m \geq 1$. What is the probability that the ice cream machine is broken at least once in m visits to McDonald's?

b) Fix an integer $m \geq 1$. Find the probability your mth visit to McDonald's is the first time that the ice cream machine is broken.

Hint: What happened on each of the first $m - 1$ visits?

c) Suppose you visit McDonald's $m = 8$ times. What is the probability that the ice cream machine is broken exactly 3 times in your 8 visits?

Problem 4. Something Fishy

Somi Somi is a dessert store that sells taiyaki, a Japanese fish-shaped waffle that contains fillings. Some example taiyaki filling flavors include custard, chocolate, and cheese. Each taiyaki has just one flavor.

The owners of Somi Somi come up with 50 different "limited-time" taiyaki flavors. They decide to hold a promotion where they give one taiyaki with a single "limited-time" flavor to each customer who comes to their store, for free. Assume that they have an unlimited supply of each "limited-time" flavor, and that each free taiyaki is equally likely to be any one of the 50 "limited-time" flavors.

a) You go to Somi Somi 15 times and collect 15 free taiyakis. What is the probability that you end up with 15 different "limited-time" flavors?

b) If you collect 15 free taiyakis, what is the probability you end up with 14 different "limited-time" flavors?

Problem 5. Prime Factorization

We've recorded a hint/walkthrough video for this problem — we highly encourage you to watch it: https://youtu.be/Jx1GrwQ44HU

Recall, a **prime number** is a natural number greater than 1 that cannot be written as a product of smaller natural numbers. For instance, 2 and 3 are prime, while 6 and 24 are not (because $6 = 2 \cdot 3$ and $24 = 2 \cdot 12 = 3 \cdot 8 = ...$). Natural numbers that are not prime (and not equal to 1) are called **composite**.

The **Fundamental Theorem of Arithmetic** states that every natural number greater than 1 can be written as the product of prime factors, and that every natural number has a unique representation in this form. (If a is a factor of b, then $\frac{b}{a}$ is a whole number.)

For example, 12 can be written as $2^2 \cdot 3$, and 1200 can be written as $2^4 \cdot 3 \cdot 5^2$.

To compute the prime factorization of a number, one strategy is to repeatedly divide by the smallest prime factor that divides that number. For instance, here's the process of prime factoring 1200 ("pf" stands for "prime factor"):

$$\begin{aligned}
1200 &= 1200 \\
&= 2 \cdot 600 \quad \text{(2 is the smallest pf of 1200)} \\
&= 2 \cdot 2 \cdot 300 \quad \text{(2 is the smallest pf of 600)} \\
&= 2 \cdot 2 \cdot 2 \cdot 150 \quad \text{(2 is the smallest pf of 300)} \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 75 \quad \text{(2 is the smallest pf of 150)} \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 25 \quad \text{(3 is the smallest pf of 75)} \\
&= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5 \quad \text{(5 is the smallest pf of 25)} \\
&= 2^4 \cdot 3 \cdot 5^2
\end{aligned}$$

A handy shortcut is that if the number we are prime factoring ends in a 0, we know that is divisible by 10 and hence we can pull out a factor of $2 \cdot 5$.

You may be thinking, **what does this have anything to do with combinatorics?** Good question — we're getting there.

Suppose we now want to determine the number of factors that 1200 has. One rather painful way to do it would be to enumerate them manually — 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, etc. Of course, there's a better way.

Each factor of 1200 will be made up of some number of 2s, some number of 3s, and some number of 5s, all multiplied together. If you don't believe this, write out a few factors of 1200, and you will see that each of them is a product of 2s, 3s, and 5s. For example:

$$\begin{aligned}
24 &= 2^3 \cdot 3 \\
30 &= 2 \cdot 3 \cdot 5 \\
120 &= 2^5 \cdot 5
\end{aligned}$$

There are 5 options for the number of 2s a factor could have: 0, 1, 2, 3 or 4 (meaning a factor of 1200 could either have a factor of 2^0, or 2^1, or 2^2, or 2^3, or 2^4). Similarly, there are 2 options for the number of 3s a factor could have (either 0 or 1) and 3 options for the number of 5s (0, 1, or 2).

In other words, each factor of 1200 will look like

$$2^a \cdot 3^b \cdot 5^c$$

where

$$0 \leq a \leq 4 \qquad 0 \leq b \leq 1 \qquad 0 \leq c \leq 2$$

Since we're making three successive choices, we take the product of the number of choices at each step, yielding

$$\text{number of factors of } 1200 = (4+1)(1+1)(2+1) = 30$$

Note that this counts both 1 ($2^0 \cdot 3^0 \cdot 5^0$) and 1200 ($2^4 \cdot 3^1 \cdot 5^2$) as factors of 1200.

a) Let's try out a few examples. Show your work.

1. Compute the prime factorization of 2500, and use it to determine the number of factors of 2500.
2. Compute the prime factorization of 7260, and use it to determine the number of factors of 7260.

In both subparts, manually listing out all factors will not receive credit; you must use the counting technique outlined in the question.

b) How many factors of 2500 are multiples of 50?

Hint: Think about how this condition restricts the number of options for each prime factor.

c) How many factors of 7260 are multiples of 2 but not multiples of 121?

Again, think about the possibilities for the exponents on each of the prime factors.