DSC 40A - Group Work Session 3

due Monday, April 22nd at 11:59PM

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **tag all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive full credit, you must work in a group of two to four students for at least 50 minutes in your assigned discussion section. You can also self-organize a group and meet outside of discussion section for 80 percent credit. You may not do the groupwork alone.

1 Objects in Linear Algebra

Problem 1.

For each subproblem, answer with one of the following choices:

- a scalar
- a vector in $\mathbb{R}^d$
- a vector in $\mathbb{R}^n$
- a $d \times d$ matrix
- a $d \times n$ matrix
- an $n \times n$ matrix
- an $n \times d$ matrix

a) For each $i = 1, ..., d$, let $\vec{x}^{(i)}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\sum_{i=1}^{d} \vec{x}^{(i)^T} \vec{x}^{(i)}$$

b) For each $i = 1, ..., d$, let $\vec{x}^{(i)}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\sum_{i=1}^{d} \vec{x}^{(i)} \vec{x}^{(i)^T}$$

c) Let $\vec{x}$ be a vector in $\mathbb{R}^n$, and let A be an $n \times n$ matrix. What type of object is:

$$\vec{x}^T A \vec{x}$$

d) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\frac{\vec{x}}{\|\vec{x}\|}$$

e) Let $\vec{x}$ be a vector in $\mathbb{R}^n$, and let A be a $d \times n$ matrix. What type of object is:

$$\frac{A\vec{x}}{\|\vec{x}\|} + (\vec{x}^T A^T A\vec{x})A\vec{x}$$

f) Let A be a $d \times n$ matrix. Suppose $A^T A$ is invertible. What type of object is:

$$(A^T A)^{-1}$$

2 Vector Projection

In Lecture 6, we started looking at how to project a vector $\vec{y}$ onto $\text{span}(\vec{x})$. Our conclusion was that the vector in $\text{span}(\vec{x})$ that was closest to $\vec{y}$ was the vector $w^*\vec{x}$, where:

$$w^* = \frac{\vec{x} \cdot \vec{y}}{\vec{x} \cdot \vec{x}}$$

Problem 2.

Show that $\vec{y} - w^*\vec{x}$ is orthogonal to $\vec{x}$. This is why $w^*\vec{x}$ is also known as the **orthogonal projection**.

Problem 3.

Vectors in $\text{span}(\vec{x})$ are all of the form $w\vec{x}$, where w is some scalar. For any w, the associated error vector is $\vec{e} = \vec{y} - w\vec{x}$.

Show that $w^*\vec{x}$ is the vector in $\text{span}(\vec{x})$ that minimizes the magnitude of the error vector. That is, show that w^* minimizes:

$$\text{error}(w) = \|\vec{y} - w\vec{x}\|$$

Hint: Note that minimizing $\|\vec{y} - w\vec{x}\|$ is equivalent to minimizing $\|\vec{y} - w\vec{x}\|^2$, and that if $\vec{v}$ is a vector, then $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$.

3 Matrix Multiplication

In Lectures 6 and 7, we will start to express the linear combination $w_1\vec{x}^{(1)} + w_2\vec{x}^{(2)} + ... + w_d\vec{x}^{(d)}$ as a matrix-vector multiplication $X\vec{w}$. Let's gain some familiarity with this idea.

Problem 4.

Let's brush up on our matrix-vector multiplication skills. Suppose we have a matrix and a vector defined as follows:

$$X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 1 & -2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

Evaluate $X\vec{w}$.

Problem 5.

Perhaps you noticed something while computing $X\vec{w}$ in the above problem. In particular, you may recall from MATH 18 that the matrix-vector multiplication, $X\vec{w}$, is a linear combination of the columns of the matrix, X, by the appropriate weights from the vector, $\vec{w}$.

Fill in each blank below with a single number using the numbers from Problem 4.

$$X\vec{w} = \underline{\quad} \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} + \underline{\quad} \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} + \underline{\quad} \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix}$$

Problem 6.

Now, let's generalize this concept. Let X be an $n \times d$ matrix, such that each column, $\vec{x}^{(i)}$ is a vector in $\mathbb{R}^n$. Let $\vec{w}$ be a vector in $\mathbb{R}^d$. Fill in the blanks:

$$X\vec{w} = \sum_{i=1}^{\square} \square$$