DSC 40A Fall 2025 - Group Work Session 1

due Monday, September 29th at 11:59PM

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **tag all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive full credit, you must work in a group of two to four students for at least 50 minutes in your assigned discussion section. You can also self-organize a group and meet outside of discussion section for 80 percent credit. You may not do the groupwork alone.

1 Objects in Linear Algebra

Problem 1.

Let $n, d \geq 1$ be fixed positive integers. For each subproblem, answer with one of the following choices:

- a scalar
- a vector in $\mathbb{R}^d$
- a vector in $\mathbb{R}^n$
- a $d \times d$ matrix
- a $d \times n$ matrix
- an $n \times n$ matrix
- an $n \times d$ matrix

a) For each $i = 1, ..., d$, let $\vec{x}^{(i)}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\sum_{i=1}^{d} \vec{x}^{(i)T} \vec{x}^{(i)}$$

Solution: A scalar.

b) For each $i = 1, ..., d$, let $\vec{x}^{(i)}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\sum_{i=1}^{d} \vec{x}^{(i)} \vec{x}^{(i)T}$$

Solution: An $n \times n$ matrix.

c) Let $\vec{x}$ be a vector in $\mathbb{R}^n$, and let A be an $n \times n$ matrix. What type of object is:

$$\vec{x}^T A \vec{x}$$

Solution: A scalar.

d) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. What type of object is:

$$\frac{\vec{x}}{\|\vec{x}\|}$$

Solution: A vector in $\mathbb{R}^n$.

e) Let $\vec{x}$ be a vector in $\mathbb{R}^n$, and let A be a $d \times n$ matrix. What type of object is:

$$\frac{A\vec{x}}{\|\vec{x}\|} + (\vec{x}^T A^T A\vec{x})A\vec{x}$$

Solution: A vector in $\mathbb{R}^d$.

f) Let A be a $d \times n$ matrix. Suppose $A^T A$ is invertible. What type of object is:

$$(A^T A)^{-1}$$

Solution: An $n \times n$ matrix.

Problem 2.

Let $p \geq 1$ be a fixed positive integer and let $x, y \in \mathbb{R}^p$ be fixed **nonzero** vectors. Writing x in column vector notation we have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}, \tag{1}$$

and similarly for y. Of the following ten expressions, five are provably equal (meaning, they are all equal to each other **regardless** of x, y). Circle them and eliminate the remaining five "odd ones out" which are not provably equal (meaning, they are not equal to the other expressions in general unless we "know" x, y).

1. xy^T.
2. $\|y\|x^T\widehat{y}$ where $\|y\|$ is the length of y and $\widehat{y}$ is the unit vector parallel to y.
3. $\sum_{i=1}^{p-1} x_i y_{i+1}$.
4. $\left(\sum_{i=1}^{p} (x_i y_i)^3\right)^{1/3}$
5. $\sum_{i=3}^{p+2} x_{i-2} y_{i-2}$.
6. $(O^T x)^T (O^T y)$, where $O \in \mathbb{R}^{p\times p}$ is an orthogonal matrix satisfying $O^T O = \mathrm{Id}_{p\times p}$.
7. $\frac{1}{2}(Ax)^T y$, where $A \in \mathbb{R}^{p\times p}$ is any matrix for which y^T is a left eigenvector with eigenvalue 2.
8. $(U^{-1}x)^T U y$ where $U \in \mathbb{R}^{p\times p}$ is any nonsingular (i.e. invertible) matrix.
9. $\frac{1}{\|x\|^2} \mathrm{Tr}\left(y^T (yx^T)x\right)$.
10. $\frac{1}{\|x\|^2} \mathrm{Tr}\left(yx^T xx^T\right)$.

Solution: Expressions 2, 5, 6, 7, and 10 are all equal to $x^T y$, i.e., the dot product of x, y.
1. This expression is a matrix multiplication of the vector x ($n \times 1$) and the transpose of the vector y ($n \times 1 \to 1 \times n$). The result will be a $n \times n$ matrix.
2. By definition of a unit vector, $\widehat{y} = \frac{\vec{y}}{\|y\|}$. Expand the equation: $\|y\| x^T \widehat{y} = \|y\| x^T \frac{\vec{y}}{\|y\|} = x^T y$. This is equivalent to the dot product.
3. The summation notation is just slightly different from the dot product. The dot product can take the form of $x_1 y_1 + x_2 y_2 + ... x_p y_p$, while this expression evaluates to $x_1 y_2 + x_2 y_3 + ... x_p y_p + 1$.
4. This one is testing your understanding on summation notation. $\left(\sum_{i=1}^{p} (x_i y_i)^3\right)^{1/3}$ is not equivalent to $\sum_{i=1}^{p} x_i y_i$, same reason as $(a+b)^2 \neq a^2 + b^2$.
5. Expand the equation, we get

$$\sum_{i=3}^{p+2} x_{i-2} \cdot y_{i-2} = x_{3-2} \cdot y_{3-2} + x_{4-2} \cdot y_{4-2} + ... x_{p+2-2} \cdot y_{p+2-2} = x_1 y_1 + x_2 y_2 + ... = x^T y.$$

6. Recall $(AB)^T = B^T A^T$. $(O^T x)^T (O^T y) = x^T O O^T y$. Given O is an orthogonal matrix, by definition, $O^T = O^{-1}$, and by the definition of invertible matrices, $O^{-1} O = O O^{-1}$, $O O^{-1} = O^T O = O O^T = Id$, the expression simplifies to $x^T y$.
7. By definition of left eigenvalue, $y^T A - 2y^T$. Transpose both sides will get us $A^T y = 2y$. Substitute this: $1/2 (Ax)^T y = 1.2 x^T A^T y = 1/2 x^T (2y) = x^T y$.
8. Simplifying the expression gets us $x^T (u^- 1)^T U) y$. even though $U^{-1} U = Id$, our expression is not in this form. We cannot simplify it further.
9. $\frac{1}{\|x\|^2} \operatorname{Tr}\left(y^T (y x^T) x\right) = \frac{1}{\|x\|^2} \operatorname{Tr}\left((y^T y)(x^T x)\right) = \frac{1}{\|x\|^2} \cdot \|x\|^2 \cdot \|y\|^2 = \|y\|^2$. The Trace function here can be ignored, because both $(y^T y)$ and $(x^T x)$ come out to be scalars, and the sum of the diagonal of a scalar value is just itself.
10. $\frac{1}{\|x\|^2} \operatorname{Tr}\left(y x^T x x^T\right) = \frac{1}{\|x\|^2} \operatorname{Tr}\left(y (x^T x) x^T\right) = \frac{1}{\|x\|^2} \cdot \|x\|^2 \operatorname{Tr}\left(y x^T\right) = Tr(y x^T)$. The matrix of $y x^T$ is:

$$\begin{bmatrix} y_1 x_1 & y_1 x_2 & \cdots & y_1 x_p \\ y_2 x_1 & y_2 x_2 & \cdots & y_2 x_p \\ \vdots & \vdots & \ddots & \vdots \\ y_p x_1 & y_p x_2 & \cdots & y_p x_p \end{bmatrix}$$

The trace is the sum of the diagonal values. We have $y_1 x_1 + y_2 x_2 + ... = x^T y$.

2 Matrix Multiplication

This is intended to help review some concepts from MATH 18 related to matrix-vector and matrix-matric multiplication.

Problem 3.

Let's brush up on our matrix-vector multiplication skills. Suppose we have a matrix and a vector defined as follows:

$$X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 1 & -2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}$$

Evaluate $X\vec{w}$.

Solution:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + 3 \cdot (-1) \\ 1 \cdot 4 + 2 \cdot 3 + 3 \cdot (-1) \\ 5 \cdot 4 + 1 \cdot 3 + (-2) \cdot (-1) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 25 \end{bmatrix}$$

Problem 4.

Perhaps you noticed something while computing $X\vec{w}$ in the above problem. In particular, you may recall from MATH 18 that the matrix-vector multiplication, $X\vec{w}$, is a linear combination of the columns of the matrix, X, by the appropriate weights from the vector, $\vec{w}$.

Fill in each blank below with a single number using the numbers from Problem 4.

$$X\vec{w} = \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} + \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix} + \_\_ \begin{bmatrix} \_\_ \\ \_\_ \\ \_\_ \end{bmatrix}$$

Solution:

$$X\vec{w} = 4 \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

Problem 5.

Now, let's generalize this concept. Let X be an $n \times d$ matrix, such that each column, $\vec{x}^{(i)}$ is a vector in $\mathbb{R}^n$. Let $\vec{w}$ be a vector in $\mathbb{R}^d$. Fill in the blanks:

$$X\vec{w} = \sum_{i=1}^{\square} \square$$

Solution:

$$X\vec{w} = \sum_{i=1}^{d} w_i \vec{x}^{(i)}$$

3 Subspaces and Bases

This section contains some practice problems to help review the concepts of vector subspaces and vector space bases.

Problem 6.

Start here: What are the three criteria for a subset S of some vector space V to qualify as a vector **subspace**?

Solution: (1) S must be nonempty, or equivalently as an axiom, $0 \in S$; (2) S must be closed under scalar multiplication; and (3), S must be closed under vector addition.

Problem 7.

For each of the following scenarios, determine whether the provided subset S qualifies as a vector subspace. *Bonus: What is the dimension of S in the case(s) where it is a vector subspace?*

a) $V = \mathbb{R}^n$ and S is the set of all $x \in V$ such that $x^T \mathbf{1}_n = 0$ where $\mathbf{1}_n$ is the vector of all ones.

Solution: This is a subspace and its dimension is $n - 1$.

b) $V = \mathbb{R}^n$ and S is the set of all $x \in V$ such that $\sum_{i=1}^n x_i = -2$.

Solution: Not a subspace - does not contain the zero vector.

c) $V = \mathbb{R}^{n \times n}$ and S is the set of all matrices $A \in V$ such that $A^2 =$ A

Solution: Not a subspace - S is not closed under scalar multiplication, e.g., $A = \mathrm{Id}_{n \times n}$ and $\lambda = 2$.

d) $V = \mathbb{R}^{n \times n}$ and S is the set of all matrices $A \in V$ such that $BA = 0_{n \times n}$ for some matrix B.

Solution: This is a subspace and its dimension is $n\text{nullity}(B)$ where $\text{nullity}(B)$ is the dimension of the kernel of B, or $n - \text{rank}(B)$. To see this, we identify A with a collection of column vectors $A = [x_1, \ldots, x_n]$ with $x_i \in \mathbb{R}^n$. Then $BA = [Bx_1, \ldots, Bx_n]$. Therefore, in order for $BA = 0$, we must have $Bx_1 = 0, Bx_2 = 0$, and so on; i.e., each x_i must belong to the kernel of B. Therefore, the collection of matrices A satisfying $BA = 0$ must be identical to n copies of $\ker(B)$.

Problem 8.

Another quick check: What are the three criteria for a subset B of some vector space V to qualify as a **basis** for V?

Solution: (1) S must be nonempty; (2) the vectors in S must be linearly independent; and (3), S must span V.

Problem 9.

Let e_i be the i-th standard basis vector which is one at index i and zero otherwise. Which of the following sets $\{x_i\}_{i=1}^n$ form a basis of $\mathbb{R}^n$?

a) $x_i = e_i - e_1$ for $1 \le i \le n$.

Solution: Not a basis, $x_1 = 0_n$, and no set containing the zero vector can also be linearly independent.

b) $x_i = e_i + \mathbf{1}_n$ for $1 \le i \le n$ where $\mathbf{1}_n$ is the vector of all ones. *Hint: Check this for case $n = 2$ and $n = 3$ to get started.*

Solution: This is a basis. For (2), if we set a linear combination of the x_i's equal to the zero vector for some constants c_i, we get

$$\sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} c_i e_i + \left(\sum_{i=1}^{n} c_i\right) \mathbf{1}_n = 0_n.$$

Thus, $\sum_{i=1}^{n} c_i e_i = -\left(\sum_{i=1}^{n} c_i\right) \mathbf{1}_n$. If one compares entries of the vectors side-by-side, we see that $c_j = -\left(\sum_i c_i\right)$ for all $1 \le j \le n$. Thus all of the c_j's are actually the same number, call it a. Then $a = -na$ for which the only solution is $a = 0$. Since $\{x_i\}$ has n linearly independent vectors in $\mathbb{R}^n$, it must also span $\mathbb{R}^n$ and (3) follows.

c) $x_i = ie_i$. In other words, $x_1 = e_1, x_2 = 2e_2, x_3 = 3e_3$, and so on.

Solution: This is a basis. Perhaps the easiest way to check is to observe that the matrix $X = [x_1 x_2 \cdots x_n]$ is the diagonal matrix

$$X = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix}$$

for which we can immediately find the inverse

$$X = \begin{bmatrix} 1 & & & \\ & 1/2 & & \\ & & \ddots & \\ & & & 1/n \end{bmatrix}$$

thus X is nonsingular, and therefore the set is a basis.