Lecture 14

More Combinatorics Examples

DSC 40A, Summer 2024

Announcements

- Homework 6 is due on **Friday at 11:59PM**.
  - Combinatorics is hard! Come to office hours.
- How are office hours working for everyone?
  - Balance of in-person/virtual? Days of the week? Morning/afternoon?

Agenda

- Example: Selecting students.
- Example: An unfair coin.
- Example: Poker.
- Example: Chess.

Remember, we've posted **many** probability resources on the resources tab of the course website. These will come in handy! Specific resources you should look at:

- The DSC 40A probability roadmap, written by Janine Tiefenbruck.
- The textbook Theory Meets Data, which explains many of the same ideas and contains more practice problems.

For combinatorics specifically, there are two supplementary videos that you should watch. Both are linked in this playlist, which is also linked at dsc40a.com.

Question 🤔

Answer at q.dsc40a.com

Remember, you can always ask questions at q.dsc40a.com!

If the direct link doesn't work, click the "🤔 Lecture Questions" link in the top right corner of dsc40a.com.

Example: Selecting students

We'll start by working through a few examples we didn't get to finish yesterday.

Summary

Suppose we want to select k elements from a group of n possible elements. The following table summarizes whether the problem involves sequences, permutations, or combinations, along with the number of relevant orderings.

	Yes, order matters	No, order doesn't matter
With replacement Repetition allowed	$\boxed{n^k}$ possible **sequences**	more complicated: watch **this video**, or see the domino example
Without replacement Repetition not allowed	$\boxed{\frac{n!}{(n-k)!}}$ possible **permutations**	$\boxed{\binom{n}{k}}$ **combinations**

Using the binomial coefficient, $\binom{n}{k}$

As an example, let's evaluate:

$$\binom{9}{2} + \binom{9}{3}$$

Combinatorics as a tool for probability

- If S is a sample space consisting of equally-likely outcomes, and A is an event, then $\mathbb{P}(A) = \frac{|A|}{|S|}$.
- In many examples, this will boil down to using permutations and/or combinations to count $|A|$ and $|S|$.
- **Tip**: Before starting a probability problem, always think about what the sample space S is!

Overview: Selecting students

We're going answer the same question using several different techniques.

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random without replacement. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 1: Using permutations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 2: Using permutations and the complement

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 3: Using combinations

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Selecting students

Method 4: The "easy" way

There are 20 students in a class. Avi is one of them. Suppose we select 5 students in the class uniformly at random **without replacement**. What is the probability that Avi is among the 5 selected students?

Question 🤔

Answer at q.dsc40a.com

With vs. without replacement

We've determined that a probability that a random sample of 5 students from a class of 20 **without replacement** contains Avi (one student in particular) is $\frac{1}{4}$.

Suppose we instead sampled **with replacement**. Would the resulting probability be equal to, greater than, or less than $\frac{1}{4}$?

- A. Equal to.
- B. Greater than.
- C. Less than.

Example: An unfair coin

Followup: An unfair coin

In the video you were asked to watch, we flipped an **unfair** coin 10 times, where the coin was biased such that for each flip, $\mathbb{P}(\text{heads}) = \frac{1}{3}$.

1. What is the probability that we see the specific sequence HHHHTTTTTT?

2. What is the probability that we see exactly 4 heads?

Followup: An unfair coin

3. What is the probability that we see exactly k heads, where $0 \leq k \leq 10$?

4. What is the probability that we see *at least* k heads, where $0 \leq k \leq 10$?

Example: Poker

Deck of cards

- There are **52** cards in a standard deck.
- Each card has a suit (4 possibilities) and a value (13 possibilities).

♥: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♦: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♣: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
♠: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A

- In poker, each player is dealt 5 cards, called a **hand**. The order of cards in a hand does not matter.

Deck of cards

1. How many 5 card hands are there in poker?

2. How many 5 card hands are there where all cards are of the same suit (i.e. a "flush")?

Deck of cards

3. How many 5 card hands are there that include four cards of the same value (i.e. a "four-of-a-kind")?

Deck of cards

4. How many 5 card hands are there that have all card values consecutive (i.e. a "straight")?

Deck of cards

5. How many 5 card hands are there that have all card values consecutive and of the same suit (i.e. a "straight flush")?

Deck of cards

6. How many 5 card hands are there that include exactly one pair of values (e.g. aabcd)?

Example: Chess

Chess pieces

(Source: Spring 2023 Midterm 2, Problem 3)

A set of chess pieces has 32 pieces. 16 of these are black and 16 of these are white. **In each color**, the 16 pieces are:

$$8 \text{ pawns, } 2 \text{ bishops, } 2 \text{ knights, } 2 \text{ rooks, } 1 \text{ queen, and } 1 \text{ king}$$

When there are multiple pieces of a given color and type (for example, 8 white pawns), we will assume they are **indistinguishable** from one another.

In this problem, a **lineup** is a way of arranging items in a straight line.

Chess pieces

1. A chess player lines up all 16 **white pieces** from the set of chess pieces. How many different-looking lineups can be created? Remember, some pieces look the same.

Chess pieces

2. A chess player lines up all 16 **pawns** from the set of chess pieces. How many lineups have white pawns on both ends?

Chess pieces

3. A chess player lines up all 16 **pawns** from the set of chess pieces. Assuming that each different-looking lineup is equally likely, what is the probability that the lineup has two of the same-colored pawns on both ends (both black or both white)?

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