DSC 40A Fall 2024 - Group Work Session 7
due Monday, Nov 18th at 11:59PM

Write your solutions to the following problems by either typing them up or handwriting them on another piece of paper. **One person** from each group should submit your solutions to Gradescope and **tag all group members** so everyone gets credit.

This worksheet won't be graded on correctness, but rather on good-faith effort. Even if you don't solve any of the problems, you should include some explanation of what you thought about and discussed, so that you can get credit for spending time on the assignment.

In order to receive full credit, you must work in a group of two to four students for at least 50 minutes in your assigned discussion section. You can also self-organize a group and meet outside of discussion section for 80 percent credit. You may not do the groupwork alone.

1 Combinatorics

In probability, when all outcomes in the sample space are equally likely, the probability of an event is the number of outcomes in the event divided by the number of outcomes in the sample space. Thus, the probability of the event reduces to two counting (or combinatorics) questions, which ask *how many* outcomes are possible. When solving a counting question, it helps to write down one example outcome, then try to think about how many options we had at each step of generating this example.

There are a few basic combinatorial objects that we've studied in this class, namely sequences, permutations, and combinations. The hard part is often determining which one to use in which situation, which comes down to two important questions:

- Does the order in which I select the objects matter? In other words, does it count as different or the same if I choose the same objects in a different order?
- Am I selecting objects with or without replacement? In other words, am I allowed to have repeated objects?

Sequences:

A sequence is obtained by selecting k elements from a group of n possible elements with replacement, such that order matters. The number of sequences is

$$n^k.$$

Permutations:

A permutation is obtained by selecting k elements from a group of n possible elements without replacement, such that order matters. The number of permutations is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

Combinations:

A combination is obtained by selecting k elements from a group of n possible elements without replacement, such that order does not matter. The number of combinations is

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!k!}.$$

Problem 1. Herb Garden

You want to plant an herb garden, so you go to a garden store that has 50 different herbs: 28 are culinary herbs, 12 are medicinal herbs, and 10 are aromatic herbs. You select 5 herbs for your herb garden by taking a random sample **without replacement** from the 50 available herbs.

a) If you consider the herbs you select as a combination (i.e. the order in which you select each herb does not matter), how many combinations of 5 herbs are possible?

b) If you consider the herbs you select as a combination (i.e. the order in which you select each herb does not matter), how many combinations of 5 herbs include 2 culinary herbs and 3 aromatic herbs?

c) If you consider the herbs you select as a permutation (i.e. the order in which you select each herb matters), how many permutations of 5 herbs are possible?

d) If you consider the herbs you select as a permutation (i.e. the order in which you select each herb matters), how many permutations of 5 herbs include 2 culinary herbs and 3 aromatic herbs?

e) What is the probability that you choose 2 culinary herbs and 3 aromatic herbs for your garden?

Problem 2. Shuffling Strings

a) How many different strings can be created by shuffling the letters of DOG?

b) How many different strings can be created by shuffling the letters of GAG?

Hint: The answer is not 6.

c) How many different strings can be created by shuffling the letters of GAAAGGGG?

Hint: How can you use combinations?

d) How many different strings can be created by shuffling the letters of AGGRAVATE?

Problem 3. Xs and Os

Let $N(a, b)$ represent the number of strings you can create out of a Xs and b Os. Explain why $N(a, b)$ satisfies each of the following:

$$N(0, b) = 1 \quad (1)$$
$$N(a, 0) = 1 \quad (2)$$
$$N(a, b) = N(a - 1, b) + N(a, b - 1) \quad \text{for } a > 0 \text{ and } b > 0. \quad (3)$$

Problem 4. Tiebreaker

To break a tie among a group of $n \geq 3$ people, you come up with the following tiebreaker: Everyone flips a coin. If one person's coin is different from all the others, that person wins, and the tie is broken! Otherwise, repeat the process.

a) What is the probability that the tie is broken after the first coin toss?

b) Fix an integer $k \geq 1$. Find the probability that the tie is broken after exactly k coin tosses?

2 Independence and Conditional Independence

Recall that two events A, B are **independent** if knowledge of one event occurring does not affect the probability of the other event occurring. There are three equivalent definitions of independence:

$$\mathbb{P}(A|B) = \mathbb{P}(A) \tag{4}$$
$$\mathbb{P}(B|A) = \mathbb{P}(B) \tag{5}$$
$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B) \tag{6}$$

Two events that are not independent are also called **dependent**.

Two events A and B are **conditionally independent given** C if

$$\mathbb{P}((A \cap B)|C) = \mathbb{P}(A|C) \times \mathbb{P}(B|C).$$

Notice the similarity between this definition and the third definition of independence given above. Conditional independence given C means that *when C occurs*, A and B are independent in that case. But they may or may not be independent in general.

Problem 5.

Let A and B be events in a sample space with $0 < \mathbb{P}(A) < 1$ and $0 < \mathbb{P}(B) < 1$. If A is a subset of B, can A and B be independent? If yes, give an example, otherwise prove why not.

Problem 6.

Consider two flips of a fair coin. The sample space is $S =$ all outcomes of 2 flips of a coin $= \{HH, HT, TH, TT\}$, where each has equal probability $\frac{1}{4}$. We define the event A as $A =$ first flip is heads $= \{HH, HT\}$, and the event B as $B =$ second flip is heads $= \{HH, TH\}$. You can verify that A and B are independent by showing $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$.

Now, suppose that the coin is not fair, and instead flips heads the first time with probability p and flips heads the second time with probability q.

Are A and B still independent?

Problem 7.

A box contains two coins: a regular coin and one fake two-headed coin ($\mathbb{P}(H) = 1$). Choose a coin at random and flip it twice. Define the following events.

- A: First flip is heads (H).
- B: Second flip is heads (H).
- C: Coin 1 (regular) has been selected.

Are A and B independent? Are A and B conditionally independent given C?

Prove your answers using the definitions of independence and conditional independence. Also explain your answers intuitively.